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Branes for Higgs phases and exact conformal field theories*

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Abstract: We consider multicenter supergravity solutions corresponding to Higgs phases of supersymmetric Yang–Mills theories with Z_N symmetric vacua. In certain energy regimes, we find a description in terms of a generalized wormhole solution that corresponds to the $SL(2,\mathbb{R})/U(1) \times SU(2)/U(1)$ exact conformal field theory. We show that U-dualities map these backgrounds to purely gravitational ones and comment on the relation to the black holes arising from intersecting D1- and D5-branes. We also discuss supersymmetric properties of the various solutions and the relation to 2-dim solitons, on flat space, of the reduced axion–dilaton gravity equations. Finally, we address the problem of understanding other supergravity solutions from the multicenter ones. As prototype examples we use rotating D3-branes and NS5- and D5-branes associated to non-Abelian duals of 4-dim hyper-Kähler metrics with $SO(3)$ isometry.

Keywords: D-branes, Conformal Field Models in String Theory, Black Holes in String Theory, Brane Dynamics in Gauge Theories.

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Contents

1. Introduction

Recent developments allow an understanding of strong coupling aspects of supersymmetric Yang–Mills theories (SYM) using string theory on backgrounds containing AdS spaces [1, 2] (see also [3]). It is desirable to have an exact description of strings in backgrounds that involve R–R fields (for recent progress, see [4]), since one can then interpolate between results at weak and strong coupling. For instance, this could provide practical tools to understand the 3/4 mismatch between the perturbative SYM and supergravity computations of the entropy of a large number of D3-branes at non-zero temperature [3], as well as for glueball-mass computations [5]. Even in the absence of R–R fields, in this context, there has appeared so far only one exact description, namely the $SU(2) \times U(1)$ WZW model Conformal Field Theory (CFT) [6, 7, 8] representing the throat of a semi-wormhole, as the near horizon geometry of NS5 branes [9, 7]. It has been conjectured to describe the ultraviolet regime of the 6-dim SYM theory with (unbroken) gauge group $SU(N)$ [10].

In this paper we show that, for the $SU(Nk)$ SYM theory in a Higgs phase, where the gauge group is broken in such a way that the vacuum has a Z_N symmetry, there is also an exact description in terms of the $SU(2)/U(1) \times SL(2,\mathbb{R})/U(1)$ coset CFT. The supergravity solution is an axionic instanton, with the geometrical interpretation of a semi-wormhole with a "fat" throat. From the point of view of the reduced equations for the 4-dim axion–dilaton gravity in the presence of two commuting isometries, it is the most general axionic instanton solution (in target space) that can be interpreted as a 2-dim soliton on flat space. Non-perturbative (in the $1/N$-expansion) corrections relate the backgrounds for the $SU(2)/U(1) \times SL(2,\mathbb{R})/U(1)$ and $SU(2) \times U(1)$ CFTs. We argue that they can be understood in terms of non-trivial configurations in the gauge theory side. We discuss the supersymmetric properties of the various solutions and show that simple U-dualities map them into purely gravitational ones. In addition, we indicate that they encode the information of the near-horizon geometry of the intersection of D1- and D5-branes and hence of the corresponding four and five dimensional black holes. Finally, we address the question whether or not various supergravity solutions correspond, in the extremal limit, to superpositions of multi-center static solutions. We give supporting evidence for this suggestion based on the examples of rotating D3-brane solutions and that of NS5 and D5 branes associated to certain non-Abelian duals of 4-dim hyper-Kähler metrics with $SO(3)$ isometries. We present the details in two appendices. We end the paper with comments and directions for future work.

2. Branes on a circle

Consider a d-dim supergravity solution corresponding to Nk parallel p-branes, which are separated into N groups, with k branes each, and have centers at $\vec{x} = \vec{x}_i$, $i = 1, 2, \ldots, N$. It is characterized by a harmonic function with respect to the $(n+2)$-dim space E^{n+2}, which is transverse to the branes

$$H_n = 1 + \sum_{i=1}^{N} \frac{ak}{|\vec{x} - \vec{x}_i|^n}\,, \qquad n = d - p - 3\,. \tag{2.1}$$

For a generic choice of vectors $\vec{x}_i$, the $SO(n+2)$ symmetry of the transverse space is broken.[1] Here we will make the simple choice that all the centers lie in a ring of radius r_0 in the plane defined by x_{n+1} and x_{n+2} and that $\vec{x}_i = (0, 0, \ldots, r_0 \cos\phi_i, r_0 \sin\phi_i)$, with $\phi_i = 2\pi i/N$. Hence the $SO(n+2)$ symmetry of the transverse space is broken to $SO(n) \times Z_N$. Since the $\vec{x}_i$'s correspond to non-zero vacuum expectation values (vev's) for the scalars, the corresponding super Yang–Mills theory is broken from $SU(kN)$ to $U(k)^N$, with the vacuum having a Z_N symmetry. Then (2.1) can be written as

$$\begin{aligned} H_n &= 1 + ak \sum_{i=0}^{N-1} \Big(r^2 + r_0^2 - 2 r_0 \rho \cos(2\pi i/N - \psi) \Big)^{-n/2} \,, \\ r^2 &= \vec{x}^2\,, \qquad x_{n+1} = \rho\cos\psi\,, \quad x_{n+2} = \rho \sin\psi\,. \end{aligned} \tag{2.2}$$

[1] The constant a may depend only on the Planck length $l_{\rm P}$, the string length l_s and the dimensionless string coupling constant g_s. For instance, for an M-theory configuration, $d = 11$ and $a \sim l_{\rm P}^6$, $l_{\rm P}^3$ for the M2-brane and for the M5-brane respectively. In string theory, $d = 10$ and $a \sim l_s^6 g_s^2$, l_s^2, $l_s^n g_s$ for fundamental strings NS1, for solitonic NS5 and Dp-branes respectively. The precise numerical factors can be found, for instance, in [11].

The finite sum in (2.2) can be computed for any n, if we know the result for $n = 1$ and $n = 2$.[2] In the limit where $N \to \infty$ we may actually replace the sum by an integral and give the result in terms of a hypergeometric function (hence neglecting winding-like contributions, see below):

$$\begin{aligned} H_n &= 1 + akN \int_0^{2\pi} \frac{d\phi}{2\pi} \left(r^2 + r_0^2 - 2r_0\rho\cos\phi\right)^{-n/2} \\ &= 1 + akN(r^2 + r_0^2 + 2r_0\rho)^{-n/2} F\left(\frac{1}{2}, \frac{n}{2}, 1, \frac{4r_0\rho}{r^2 + r_0^2 + 2r_0\rho}\right) . \end{aligned} \tag{2.3}$$

We expect that far away from the ring the solution reduces to that for kN branes in the origin. Indeed, we find

$$H_n \approx 1 + \frac{akN}{r^n} + \mathcal{O}\left(\frac{1}{r^{2n}}\right) , \qquad \text{for } r \gg r_0 . \tag{2.4}$$

Also we expect that the solution, very close to the ring, should be given by the single-center one smeared out completely along a transverse direction [12]. In other words our multicenter harmonic in E^{n+2} should reduce to a single-center harmonic in E^{n+1}. We let

$$x_i = \epsilon y_i, \quad i = 1, 2, \dots, n , \qquad x_{n+1} = r_0 + \epsilon y_{n+1}, \qquad x_{n+2} = \epsilon y_{n+2} , \tag{2.5}$$

where ϵ is a dimensionless parameter, which can be related to the natural scales in the theory, as we shall see in specific examples below. Indeed, we then obtain

$$\begin{aligned} H_n &\approx 1 + \frac{akN\Gamma(\frac{n-1}{2})}{2\sqrt{\pi}\Gamma(\frac{n}{2})} \frac{1}{\epsilon^{n-1} r_0 |\vec{y}|^{n-1}} , \qquad \text{as } \epsilon \to 0 , \\ \vec{y}^2 &= y_1^2 + y_2^2 + \cdots + y_{n+1}^2 . \end{aligned} \tag{2.6}$$

Hence, our general solution interpolates between the two extreme cases (2.4) and (2.6).

2.1 D5's and NS5's on a circle

In the case of D5- and NS5-branes we have $n = 2$. Then (2.2) may be computed explicitly[3]

$$\begin{aligned} H_2 &= 1 + \frac{kNl_s^2 g_s^\delta}{2r_0\rho\sinh x} \Lambda_N(x, \psi) , \\ e^x &\equiv \frac{r^2 + r_0^2}{2r_0\rho} + \sqrt{\left(\frac{r^2 + r_0^2}{2r_0\rho}\right)^2 - 1} , \end{aligned} \tag{2.7}$$

[2]Let $h_n(\lambda) \equiv \sum_{i=0}^{N-1} \left(r^2 + r_0^2 + \lambda - 2r_0\rho\cos(2\pi i/N - \psi)\right)^{-n/2}$. Then using the recursion relation $h_{n+2}(\lambda) = -\frac{2}{n}\frac{dh_n(\lambda)}{d\lambda}$ and $H_n = 1 + akh_n(0)$ we see that $h_1(\lambda)$ and $h_2(\lambda)$ are generating functions for all H_n's.

[3]It is a rather standard result of complex analysis that the infinite sum $\sum_{i=-\infty}^{+\infty} F(i)$ equals the sum of residues of the complex function $-\cot(\pi z)F(z)$ at the poles of $F(z)$ (under some assumptions on the behaviour of $F(z)$ in the complex plane). In our case the sum is finite, but with some appropriate limiting procedure the result just stated can be used.

where $\delta = 1$ (0) for D5 (NS5) branes and

$$\Lambda_N(x,\psi) \equiv \frac{\sinh(Nx)}{\cosh(Nx) - \cos(N\psi)} \,. \tag{2.8}$$

Note the explicit Z_N invariance under shifts of $\psi \to \psi + \frac{2\pi}{N}$.

Now we specialize to the case of kN D5-branes on a circle of radius r_0 in the decoupling limit

$$\begin{aligned} u_i &= \frac{x_i}{l_s^2} = \text{fixed}\,, \qquad U^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2\,, \qquad u^2 = u_3^2 + u_4^2\,, \\ g_{\rm YM}^2 &= g_s l_s^2 = \text{fixed}\,, \qquad U_0 = \frac{r_0}{l_s^2} = \text{fixed}\,, \qquad l_s \to 0\,. \end{aligned} \tag{2.9}$$

We may take $r_0 \sim l_s/g_s^{1/2}$ or $U_0 \sim 1/g_{\rm YM}$ since the coupling constant $g_{\rm YM}$ is the only scale in the classical theory. In this limit the appropriate supergravity solution is (we omit the R–R 3-form magnetic field strength)

$$\begin{aligned} \frac{1}{l_s^2} ds^2 &= \frac{V}{\sqrt{g_{\rm YM}^2 N k}}\, ds^2(E^{1,5}) + \frac{\sqrt{g_{\rm YM}^2 N k}}{V}\, du_i du_i\,, \\ e^{2\Phi} &= \frac{g_{\rm YM}^2 V^2}{Nk}\,, \end{aligned} \tag{2.10}$$

where V is a function of U and u defined as

$$\begin{aligned} V(U,u) &= \left(\left(U^2 + U_0^2\right)^2 - 4U_0^2 u^2\right)^{1/4} \Lambda_N^{-1/2}(x,\psi)\,, \\ e^x &\equiv \frac{U^2 + U_0^2}{2U_0 u} + \sqrt{\left(\frac{U^2+U_0^2}{2U_0 u}\right)^2 - 1}\,. \end{aligned} \tag{2.11}$$

Note that e^x has the same form as in (2.7). The first factor in the expression for $V(U,u)$ is what we would have obtained had we used (2.3), i.e. when $Nx \gg 1$.[4] The analysis which description is valid, the supergravity or the "perturbative" 6-dim SYM theory one, parallels the one performed in [1, 10] for the one-center solution or equivalently when we keep all the branes well below substringy distances (at $r \approx 0$). The scalar curvature for the metric in (2.10) is

$$R = -\frac{12}{\sqrt{g_{\rm YM}^2 N k}} \frac{U^2}{V^3} + \mathcal{O}\left(e^{-Nx}\right)\,, \tag{2.12}$$

and therefore the supergravity approximation is valid when it is small. In the opposite limit the SYM picture should be trusted. String loop corrections are controlled by

[4] The generating function $h_2(\lambda)$ as defined in footnote 1 can be read off eq. (2.7), but in the expression for e^x one should replace r^2 by $r^2 + \lambda$.

the string coupling e^{Φ}, in (2.10). When this becomes strong, we should pass to the S-dual description in terms of kN NS5-branes with a supergravity description that uses the same harmonic function. The corresponding metric, antisymmetric tensor field strength and dilaton are

$$\begin{aligned} \frac{1}{l_s^2} ds^2 &= ds^2(E^{1,5}) + NkV^{-2}\, du_i du_i\,, \\ \frac{1}{l_s^2} H_{ijk} &= Nk\epsilon_{ijkl}\partial_l V^{-2}\,, \\ e^{2\Phi} &= \frac{Nk}{g_{\rm YM}^2 V^2}\,, \end{aligned} \tag{2.13}$$

and represent an axionic instanton [7]. The scalar curvature for the metric in (2.13) is

$$R = \frac{6}{Nk}\frac{U^2}{V^2} \ + \ \mathcal{O}\left(e^{-Nx}\right). \tag{2.14}$$

Note that for energy regimes $g_{\rm YM}U - 1 \geq \frac{1}{N}$, the factor Λ_N in the expression for $V(U,u)$ can be ignored and be set to 1. Using the above, and assuming that $N \gg 1$ and that $U_0 \sim 1/g_{\rm YM}$, we find that the SYM description is valid for energies close to $U = U_0 \sim 1/g_{\rm YM}$ in the range $\frac{1}{N} \ll g_{\rm YM}U - 1 \ll \frac{1}{N^{1/3}}$. For $g_{\rm YM}U \ll 1$ and for $1 \ll g_{\rm YM}U \ll \sqrt{N}$, we should use the solution for D5-branes and for $g_{\rm YM}U \gg \sqrt{N}$ the one for NS5-branes. For the latter case, we show below that there exists an exact description that allow us to go beyond the supergravity approximation. For energy regimes $g_{\rm YM}U - 1 \leq \frac{1}{N}$ the function Λ_N can no longer be ignored in the analysis. It can be shown that the SYM description is then valid, but for $0 \leq g_{\rm YM}U - 1 \ll \frac{1}{N}$ the gauge group should be an unbroken $U(k)$ instead of the broken $SU(Nk) \to U(k)^N$.

We may easily show that

$$\Lambda_N = \frac{1}{2}\left(\coth\left(N(x+i\psi)\right) + \coth\left(N(x-i\psi)\right)\right) = 1 + \sum_{m\neq 0} e^{-N(|m|x - im\psi)}\,. \tag{2.15}$$

Hence, Λ_N is a harmonic function in the (x,ψ)-plane. Also, in the $\frac{1}{N}$-expansion, Λ_N has only a "tree-level" contribution, whereas the rest of the terms in the infinite sum are non-perturbative. In particular, the exponential factors $N(|m|x - m\psi)$ are likely to originate from configurations of the 6-dim spontaneously broken gauge theory that interpolate between the N different degenerate vacua. The same exponentials, but with different coefficients in the infinite sum (2.15), appear for the case of D3-branes (see (3.1) below) and also for D1-branes and the D(-1) instantons. Finding an interpretation in terms of configurations in the $\mathcal{N}=4$ spontaneously broken SYM theory is important. In that respect, we note the recent work on the identification of D3-branes in the bulk of $AdS_5 \times S^5$ with 4-dim $\mathcal{N}=4$ $SU(N)$ SYM (for large N) in the Coulomb branch, where Higgs vev's are given to the scalar fields [13] (for related work see also [14]). A similar problem associated with quantum corrections to the moduli space for hypermultiplets near a conifold singularity was addressed in [15]. We hope to report work along these lines in the future.

2.1.1 Semi-wormhole and solitonic interpretation

For the case of NS5-branes on the circle the non-trivial 4-dim part of the background has the form of an axionic instanton

$$\begin{aligned} ds^2 &= H_2 dx_i dx_i \,, \qquad i = 1,2,3,4\,, \\ H_{ijk} &= \epsilon_{ijkl}\partial_l H_2\,, \\ e^{2\Phi} &= H_2\,, \end{aligned} \tag{2.16}$$

with the harmonic function given by (2.7). In the region where $Nx \gg 1$, it becomes

$$H_2 \approx 1 + l_s^2 kN \left(\left(r^2 + r_0^2\right)^2 - 4r_0^2\rho^2\right)^{-1/2} + \mathcal{O}\left(e^{-Nx}\right), \tag{2.17}$$

which corresponds to (2.3) for $n = 2$. Then the solution reduces to the one discussed, in a different context, in [16]. The geometrical interpretation, in that limit, is that of a semi-wormhole with a fat throat and S^3-radius $\sqrt{Nk}l_s$. However, as we get closer to any one of the centers, the solution tends to represent the throat of a wormhole with S^3-radius $\sqrt{k}l_s$. Hence, we think of (2.16) as a superposition of "microscopic" semi-wormholes distributed around a circle. This is to be contrasted with the zero size throat of the usual $SU(2) \times U(1)$ semi-wormhole, to which (2.16) reduces for $r \gg r_0$ (see (2.4)). The latter solution was given [17] an interpretation as a 2-dim soliton (on flat space) of the reduced β-functions equations in the presence of two commuting isometries. It can be shown that (2.16), with (2.17), is the most general axionic instanton solution (in target space) with two commuting isometries that has the interpretation of a soliton on flat space. This becomes apparent if one compares the expressions for $e^{2\Phi}$ given by (2.17) and by eq. (5.14) of [17]. The identification of parameters is: $M = \frac{1}{2}l_s^2 kN$ and $C_0^{(1)} = -\frac{1}{2}r_0^2$.

2.1.2 Exact conformal field theory description

In the case of NS5-branes, we may find an exact CFT description for the background (2.13) in two limiting cases. For $N = 1$, corresponding to k NS5-branes at a single point, it is known that the exact description is in terms of the $SU(2)_k \times U(1)_Q$ WZW model, where $Q = \sqrt{\frac{2}{k+2}}$ is the background charge associated with the $U(1)$ factor [6, 7, 8]. We will show that another CFT provides an exact description when $N \gg 1$. As we shall see, these two CFTs are not related by marginal deformations since they have different central charges.

A change of variables [16]

$$\begin{aligned} u_1 &= r_0 \sinh\rho \cos\theta \cos\tau\,, \qquad & u_2 &= r_0 \sinh\rho \cos\theta \sin\tau\,, \\ u_3 &= r_0 \cosh\rho \sin\theta \cos\psi, \qquad & u_4 &= r_0 \cosh\rho \sin\theta \sin\psi\,, \end{aligned} \tag{2.18}$$

transforms (2.13) to (we will omit the trivial part of the metric $ds^2(E^{1,5})$ and denote its non-trivial transverse part by $ds^2_\perp$)

$$
\begin{aligned}
\frac{1}{Nk} ds^2_\perp &= \Lambda_N(x,\psi)\left(d\rho^2 + d\theta^2 + \frac{1}{1+\tanh^2\rho\tan^2\theta}\left(\tan^2\theta\, d\psi^2 + \tanh^2\rho\, d\tau^2\right)\right)\,, \\
\frac{1}{Nk} B_{\tau\psi} &= \frac{\Lambda_N(x,\psi)}{1+\tanh^2\rho\tan^2\theta}\,, \\
\frac{1}{Nk} B_{\tau\theta} &= \frac{\cot\theta\sin(N\psi)}{\sinh(Nx)}\,\Lambda_N(x,\psi)\,, \\
e^{-2\Phi} &= \frac{g^2_{\rm YM}}{Nk} U_0^2 \Lambda_N^{-1}(x,\psi)\left(\cos^2\theta\cosh^2\rho + \sin^2\theta\sinh^2\rho\right)\,,
\end{aligned}
\tag{2.19}
$$

where $e^x = \frac{\cosh\rho}{\sin\theta}$. Note that we have not included the overall factor l_s^2, since it drops out of the σ-model as well as of the supergravity action. String perturbation theory is defined in terms of the effective dimensionless coupling $\frac{1}{Nk}$. However, as we have discussed, the background already contains non-perturbative contributions with respect to that coupling, i.e. in the expression for Λ_N.

Let us perform a T-duality transformation with respect to the vector field $\partial/\partial\tau$. Since the solution contains only NS–NS fields, the usual Buscher rules apply. We obtain a solution of type IIA supergravity, with the same six flat directions as in (2.13), and a non-trivial transverse part given by

$$
\begin{aligned}
\frac{1}{Nk} ds^2_\perp = {} & \Lambda_N\left(d\rho^2 + \coth^2\rho d\psi^2\right) + \Lambda_N^{-1}\left(\coth^2\rho + \tan^2\theta\right) d\tau^2 + \\
& + \Lambda_N\left(1 + \left(1+\coth^2\rho\cot^2\theta\right)\frac{\sin^2(N\psi)}{\sinh^2(Nx)}\right) d\theta^2 + \ 2\coth^2\rho\, d\tau d\psi + \\
& + 2\cot\theta\frac{\sin(N\psi)}{\sinh(Nx)}\left(\Lambda_N\coth^2\rho d\psi + \left(\coth^2\rho + \tan^2\theta\right) d\tau\right)\ d\theta\,, \\
e^{-2\Phi} = {} & \frac{g^2_{\rm YM} U_0^2}{Nk}\ \cos^2\theta\sinh^2\rho\,,
\end{aligned}
\tag{2.20}
$$

and zero antisymmetric tensor. In the limit $N \gg 1$, we obtain

$$
\begin{aligned}
\frac{1}{Nk} ds^2_\perp &= d\theta^2 + \tan^2\theta\ d\varphi^2 + d\rho^2 + \coth^2\rho\ d\omega^2\,, \\
e^{-2\Phi} &= \frac{g^2_{\rm YM} U_0^2}{Nk}\ \cos^2\theta\sinh^2\rho\,,
\end{aligned}
\tag{2.21}
$$

where $\omega = \tau + \psi$ and $\varphi = \tau$. This is the background corresponding to the exact CFT $SU(2)_{kN}/U(1) \times SL(2,\mathbb{R})_{kN+4}/U(1)$. In the opposite extreme case of $N = 1$, it can be shown that (2.20) reduces to

$$
\begin{aligned}
\frac{1}{k} ds^2_\perp &= d\theta^2 + \tan^2\theta\ d\varphi^2 + d\rho^2 + d\omega^2\,, \\
e^{-2\Phi} &= \frac{g^2_{\rm YM} U_0^2}{4k}\ \cos^2\theta\ e^{2\rho}\,,
\end{aligned}
\tag{2.22}
$$

which is the background for the exact CFT $SU(2)_k/U(1) \times U(1)_R \times U(1)_Q$. Here $R = 2k$ denotes the compactification radius of the bosonic field ω and $Q = \sqrt{\frac{2}{k+2}}$ is the background charge of the bosonic field ρ.[5] This is no surprise, since the backgrounds for $SU(2)_k/U(1) \times U(1)_R$ and $SU(2)_k$ are T-duality related [19].

We would like to comment briefly on the supersymmetric properties of the various solutions we have presented. As any axionic instanton, the solution (2.13) (or equivalently (2.19)) preserves half the supersymmetries of flat space. For the limiting cases $N \gg 1$ and $N = 1$, the Killing spinors for space-time supersymmetry were computed in [16]. Moreover, from the world-sheet point of view there is, in general, $\mathcal{N} = 4$ supersymmetry with three complex structures given explicitly in [16]. The background (2.20), obtained after the T-duality was performed, still has the same amount of supersymmetry, albeit part of it is realized non-locally (for details, we refer the reader to [20, 21, 16]). The reason is that the vector field $\partial/\partial\tau$, which respect to which the T-duality transformation was performed, is of the rotational type. In particular, for the case of world-sheet supersymmetry, the $\mathcal{N} = 2$ part is still locally realized. This corresponds to the ordinary $\mathcal{N} = 1$ supersymmetry enhanced to an $\mathcal{N} = 2$ using the complex structure which is a singlet of the duality group $U(1)$. However, the rest of $\mathcal{N} = 4$, corresponding to the two complex structures that form a $U(1)$ doublet, is realized by using parafermionic variables [20]. The explicit expressions for the cases of the backgrounds (2.21) and (2.22) were given in [16, 20], but similar expressions can be found for the more general background (2.20).

2.1.3 Relation to pure gravity and black holes

Let us consider a solution of type IIB supergravity, obtained by tensoring (2.20) with the 6-dim Minkowski space-time, where we compactify two of the five space-like dimensions, i.e. x_4 and x_5, on a 2-torus. By performing an S-duality and then two T-dualities along x_4 and x_5, we obtain again a solution of type IIB supergravity;this however, is purely gravitational, with metric

$$\begin{aligned} ds^2 &= f^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + f^{1/2}\left(f^{-1}ds_\perp^2 + dx_4^2 + dx_5^2\right) , \\ f &= \frac{1}{\cos^2\theta \sinh^2\rho} , \end{aligned} \tag{2.23}$$

where $ds_\perp^2$ is given by the metric in (2.20). The dilaton takes the constant value $e^{-2\Phi} = \frac{Nk}{g_{\rm YM}^2 U_0^2}$. The solution (2.23) has no apparent supersymmetry, although this is expected in a string theoretical context. It is, however, not known how to trace its supersymmetric properties from those of the original background (2.20), since it was necessary to perform an S-duality transformation in order to obtain it. Resolving this issue is still an open problem.

[5] The various shifts at the levels of the current algebras in the coset CFTs and in the background charge Q are necessary for supersymmetry to hold at the quantum level [18, 8].

In a certain sense, (2.23) is the master background from which interesting black-hole solutions can be derived. Consider the analytic continuation $t \to ix_0$ and $\omega \to it$, where we assume that $ds^2_\perp$ in (2.23) is given by the corresponding expression in (2.21). Then, using the same T- and S-dualities we described before, we obtain the Minkowski background for $E^6 \times SU(2)/U(1) \times SL(2,\mathbb{R})/SO(1,1)$. As we have mentioned, the backgrounds for the $SU(2)/U(1) \times U(1)$ coset model and the $SU(2)$ WZW model are related by an appropriate T-duality [19], and similarly for the $SL(2,\mathbb{R})/SO(1,1) \times U(1)$ coset model and the $SL(2,\mathbb{R})$ WZW model. Using these relations, we obtain the background for $E^6 \times SU(2) \times SL(2,\mathbb{R})$. This correspond to the near-horizon geometry of the intersection of NS1- and NS5-branes (or of their S-dual D1- and D5-branes). After an identification of new periodic variables in $SL(2,\mathbb{R})$ we obtain the BTZ black-hole solution with non-zero angular momentum [22]. This is related by a set of T- and S-dualities to the background of type II supergravity representing a non-extremal intersection of NS1- and NS5-branes (or of their S-dual D1 and D5) with a wave along a common direction [23] (see also [24]). The toroidal compactification of this solution to five dimensions is a non-extremal black hole [25, 11]. It can be easily seen that the four parameters characterizing this solution are preserved in the process of dualizing either by appearing explicitly in the backgrounds or by entering in the compactifications radii [23]. Black holes in four dimensions can also be discussed in a similar fashion.

3. Final comments and some open problems

It is quite natural to expect that various BPS supergravity solutions could correspond to superpositions of static brane solutions, i.e. a sum of δ-functions distribution, in some special limit. Finding the discrete distribution and not just its continuous limit[6] is important since that would correspond, in the SYM theory side, to finding the distribution of eigenvalues of the vev's of the scalar fields in the Coulomb branch, i.e. of the moduli space. Some preliminary work shows that this is the case in two classes of examples. The first one is a generalization of the static D3-brane solution of type-IIB supergravity which has, in addition to the charge and mass, angular momentum [26] (based on work in [27]). The second example corresponds to NS5-branes of type II and heterotic string theory whose non-trivial 4-dim part is described by the non-Abelian dual of 4-dim hyper-Kähler metrics with $SO(3)$ isometry. We refer to the two appendices for the details. A related question is whether or not there exists a non-BPS version of general backgrounds, with harmonic functions given by (2.7) or even (2.2). The attractive force between the branes renders their configuration on the circle unstable. According to the results of appendix A, it might

[6]If such a solution behaves as $1/r^n$ at infinity, then it surely corresponds to a continuous distribution of branes, since the basic static brane solution coincides with the Green function in E^{n+2}.

be possible to stabilize them by introducing angular momentum. If this turns out to be the case, it would be interesting to identify, in the limiting case $N \gg 1$, where the exact CFT is known, the marginal deformation that breaks supersymmetry.

In [28] the heavy quark–antiquark potential for the unbroken $\mathcal{N} = 4$ SYM in the large-N limit was computed using the AdS/CFT correspondence. The authors of [29] generalized this computation to the case when $SU(N)$ breaks to $SU(N/2) \times SU(N/2)$ by separating the branes into two groups. This corresponds in our notation to taking $N = 2$ and k general and large. They found that there are some geodesics with "confining" behaviour, i.e. giving rise to a linear potential, even though the theory is conformal and hence not expected to be confining. However, the authors demonstrated that these geodesics were unstable, even classically. Further increasing the number of centers might result in a stabilization of these trajectories. Using (2.7), (2.15) and footnotes 2 and 4, we find that the corresponding supergravity-solution harmonic function is

$$
\begin{aligned}
H_4 &= 1 + \frac{4\pi N k g_s l_s^4 (r^2 + r_0^2)}{\left((r^2 + r_0^2)^2 - 4 r_0^2 \rho^2\right)^{3/2}}\, \Sigma_N \,, \\
\Sigma_N &\equiv 1 + \sum_{m \neq 0} \left(1 + \frac{\left((r^2 + r_0^2)^2 - 4 r_0^2 \rho^2\right)^{1/2}}{r^2 + r_0^2}\, N|m| \right) e^{-N(|m|x - im\psi)} \,,
\end{aligned}
\tag{3.1}
$$

all definitions being given in the text. Properties of this harmonic function can be used to investigate how stable the "confining" behaviour is in the general case. In particular it will be interesting to study the large-N limit, where Z_N becomes a $U(1)$.

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A. Rotating branes from static ones

A rotating D3-brane solution of type-IIB supergravity was found in [26] (see also [27]). The dilaton is constant and the metric reads (we omit the self-dual 5-form):

$$
\begin{aligned}
ds^2 = {} & H^{-1/2} \left(-f\, dt^2 + dy_1^2 + dy_2^2 + dy_3^2 \right) + \\
& + H^{1/2} \left(\frac{dr^2}{f_1} + r^2 \left(\Delta d\theta^2 + \Delta_1 \sin^2\theta \; d\phi^2 + \cos^2\theta \; d\Omega_3^2 \right) - \frac{4ml \cosh\alpha}{r^4 \Delta H} \sin^2\theta dt d\phi \right),
\end{aligned}
\tag{A.1}
$$

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where

$$
\begin{aligned}
H &= 1 + \frac{2m\sinh^2\alpha}{r^4\Delta}\,, \qquad \Delta = 1 + \frac{l^2\cos^2\theta}{r^2}\,, \qquad \Delta_1 = 1 + \frac{l^2}{r^2} + \frac{2ml^2\sin^2\theta}{r^6\Delta H}\,, \\
f &= 1 - \frac{2m}{r^4\Delta}\,, \qquad f_1 = \frac{1}{\Delta}\left(1 + \frac{l^2}{r^2} - \frac{2m}{r^4}\right), \qquad\qquad (A.2)
\end{aligned}
$$

where l is the angular-momentum parameter and $\sinh^2\alpha = \sqrt{(2\pi g_s N l_s^4/m)^2 + 1/4} - \frac{1}{2}$. The extreme limit is obtained by letting $m \to 0$. After some appropriate change of variables one finds [26]

$$
\begin{aligned}
ds^2 &= H_0^{-1/2}\left(-dt^2 + dy_1^2 + dy_2^2 + dy_3^2\right) + H_0^{1/2}(dx_1^2 + \cdots + dx_6^2)\,, \\
H_0 &= 1 + \frac{8\pi g_s l_s^4 N}{\sqrt{(r^2+l^2)^2 - 4l^2\rho^2}\left(r^2 - l^2 + \sqrt{(r^2+l^2)^2 - 4l^2\rho^2}\right)}\,, \qquad\qquad (A.3) \\
r^2 &= x_1^2 + \cdots + x_6^2\,, \qquad \rho^2 = x_5^2 + x_6^2\,.
\end{aligned}
$$

The harmonic function H_0 becomes singular in the x_5-x_6 plane inside a disc of radius $r = \rho = l$.

We would like to interpret (A.3) as some superposition of N static D3-branes, other than that of N coinciding rotating D3-branes in the extremal limit. Consider N branes distributed, uniformly in the angular direction, inside a disc of radius l in the x_5-x_6 plane. Their centers are given by

$$
\begin{aligned}
\vec{x}_{ij} &= (0,0,0,0,r_{0j}\cos\phi_i, r_{0j}\sin\phi_i)\,, \\
\phi_i &= \frac{2\pi i}{N}\,, \qquad r_{0j} = l\left(j/\sqrt{N}\right)^{1/2}\,, \qquad i,j = 0,1,\ldots,\sqrt{N}-1\,. \qquad\qquad (A.4)
\end{aligned}
$$

Since we are mainly interested in the large-N limit we may take $\sqrt{N}$ = integer without loss of generality. Then, the corresponding harmonic function becomes

$$
\begin{aligned}
H_0 &= 1 + 4\pi g_s l_s^4 \sum_{i,j=0}^{\sqrt{N}-1} \frac{1}{\left(r^2 + r_{0j}^2 - 2\rho r_{0j}\cos(\phi_i - \psi)\right)^2} \\
&\approx 1 + 4\pi N g_s l_s^4 \int_0^l \frac{2r_0 dr_0}{l^2}\int_0^{2\pi}\frac{d\phi}{2\pi}\frac{1}{\left(r^2 + r_0^2 - 2\rho r_0\cos\phi\right)^2} \\
&= 1 + 4\pi N g_s l_s^4 \int_0^l \frac{2r_0 dr_0}{l^2}\frac{r^2 + r_0^2}{\left((r^2+r_0^2)^2 - 4r_0^2\rho^2\right)^{3/2}} \qquad\qquad (A.5) \\
&= 1 + 2\pi N g_s l_s^4 \frac{1}{l^2(r^2-\rho^2)}\left(1 + \frac{l^2 - r^2}{\sqrt{(r^2+l^2)^2 - 4\rho^2 l^2}}\right),
\end{aligned}
$$

where the second line is an approximation, valid for large N. It is easily seen that the last line in (A.5) equals the harmonic in (A.3). A priori it is not obvious that there exists a non-extremal version of (A.3) with H_0 given by the first line in (A.5), since non-BPS branes exert forces against one another. In the continuum limit, such an non-extremal solution exists and is given by (A.2). It that case the gravitational attraction, which is no longer balanced by just the R–R repulsion, is now balanced by forces due to the angular momentum. It would be interesting to find an analogue of this in the general case.

B. Branes and non-Abelian duality

Consider 4-dim hyper-Kähler metrics with $SO(3)$ isometry [30]. We will construct NS5-branes of type II and heterotic string theory whose non-trivial 4-dim transverse part will be the non-Abelian duals, of a particular class of these metrics, with respect to the $SO(3)$ group. In the case of type IIB, we may also consider the corresponding solution for D5-branes obtained by S-duality.

The non-Abelian dual background to 4-dim hyper-Kähler metrics with $SO(3)$ isometry is [31, 32]

$$
\begin{aligned}
ds^2 &= f^2 dt^2 + e^{2\Phi}\left((\chi\cdot d\chi)^2 + 4f^2 \sum_{k=1}^{3} \frac{1}{a_k^2} d\chi_k^2\right) , \\
B_{ij} &= e^{2\Phi} \sum_{k=1}^{3} \epsilon_{ijk}\chi_k a_k^2 \ , \\
e^{-2\Phi} &= 4\left(4f^2 + \sum_{k=1}^{3} a_k^2 \chi_k^2\right) ,
\end{aligned}
\tag{B.1}
$$

where $f = \frac{1}{2}a_1 a_2 a_3$. The functions $a_i(t)$ satisfy the first-order differential equations [30]

$$
\frac{a_i'}{a_i} = \frac{1}{2}\vec{a}^2 - a_i^2 - 2f\frac{\lambda_i}{a_i} \, , \qquad i = 1,2,3 \, . \tag{B.2}
$$

There are two distinct categories of solutions to (B.2), depending on the values of the parameters $\lambda_1, \lambda_2, \lambda_3$. The first corresponds to $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and contains the non-Abelian duals of the Taub–NUT and Atiyah–Hitchin metrics. In that case supersymmetry is realized non-locally [32]. The other case of interest to us, which corresponds to $\lambda_1 = \lambda_2 = \lambda_2 = 0$, contains the non-Abelian duals of the Eguchi–Hanson metric and is supersymmetric in the usual sense. It was also noted in [32] that the metric in (B.1) is then conformally flat. The explicit coordinate transformation, which makes the conformal flatness of the metric manifest, is

$$
x_i = 2f\frac{\chi_i}{a_i} \, , \qquad x_4 = \frac{1}{2}\vec{\chi}^2 - 4\int^t f^2(t')dt' \, . \tag{B.3}
$$

Then (B.1) is transformed into the form of an axionic instanton (2.16), with the harmonic function H given by

$$H^{-1} = 4\left(4f^2(t) + \frac{1}{4f^2(t)}\sum_{k=1}^{3} a_k^4(t)x_k^2\right), \tag{B.4}$$

where t is determined in terms of (x_i, x_4) by solving the equation

$$x_4 + 4\int^t f^2(t')dt' - \frac{1}{8f^2(t)}\sum_{k=1}^{3} a_k^2(t)x_k^2 = 0\,. \tag{B.5}$$

We have mentioned that this particular axionic instanton can be used for the construction of supergravity solutions for NS5-branes. These are distributed along the surface, in general 3-dimensional, where H in (B.4) becomes singular. In type IIB we may also consider the corresponding solution for D5-branes obtained by S-duality. It is not obvious that (B.4) corresponds to a continuous limit of a multicenter harmonic in general. However, this is the case when $a_i(t) = (-t)^{-1/2}$. Then (2.16) with (B.4) corresponds to the non-Abelian dual of flat 4-dim space with respect to the left (or right) action of the $SO(3)$ subgroup of isometries. Then (B.5) can be solved and gives $t = -(r_4 - x_4)^{-1/2}$. Substituting this in (B.4), we obtain $H^{-1} = 8r_4\sqrt{r_4 - x_4}$. This harmonic has a Dirac-string type singularity along the positive x_4-axis. It can be shown that it corresponds to the continuum limit of the sum

$$\sum_{i=0}^{N} \frac{1/\sqrt{N}}{x_1^2 + x_2^2 + x_3^2 + (x_4 - i^2/N)^2} \approx \frac{\pi}{2\sqrt{2}}\frac{1}{r_4\sqrt{r_4 - x_4}}\,. \tag{B.6}$$

It is not clear that a non-extremal version of the solution we have just discussed exists. One should try to balance the attractive force between the branes by some rotation around an axis perpendicular to x_4, say the x_1-axis.

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